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Waves generated in the configuration of a magnetically confined and field-permeated axisymmetric jet

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A cylindrical MGD configuration comprising an electrically conducting, infinitely extended (cylindrical) jet which is both permeated and contained by stream-aligned magnetic fields is considered. Axisymmetric perturbations are then generated by exposure to a stationary, weak, azimuthal current source. Ignoring relativistic and dissipative effects, asymptotic solutions complying with a radiation condition are established for the steady state. Generally speaking, the disturbed system propagates a discrete superposition of wave functions, each of which is uniquely obtained for either the upstream or downstream region. In the non-trivial cases, there is no axial wave attenuation. However, transverse attenuations are experienced within the fluid-free enveloping field. The various jet-flow regimes are thoroughly examined. It is found, in particular, that a single stationary wave is produced upstream whenever $M < 1$, $M^2+A^2 \geqslant 2$ and $\lambda\beta < 1$, while if $M > 1$ and $A > 1$ an infinity of discrete stationary waves occurs downstream, M and A being, respectively, the Mach and Alfvén numbers of the jet; $\lambda\beta$ is a parameter involving M, A and A_0 (an interface Alfvén number).

1. Introduction

Stix (1957, 1958, 1962) has demonstrated the formation of small amplitude Alfvén and ion cyclotron waves during steady harmonic excitations, induced by various current-sheet devices, of a circular cylinder of cold, pressureless, perfectly conducting plasma with zero electron mass contained axially by an external vacuum field. Bounded plasma configurations of this type, and more general configurations, are covered by the celebrated Kruskal–Schwarzschild problem (Kruskal & Schwarzschild 1954), which was primarily intended to deal with certain instability aspects. Rigorous stability analysis of a bounded plasma system with cylindrical geometry, in particular, a constricted discharge of an ionized gas, has been accomplished by Tayler (1957).

The associated stable jet problem of a streaming, magnetically sandwiched, infinite rectangular layer of plasma, which is free from an internally trapped magnetic field, was later attempted, on the linearized basis, by Savage (1967). He showed that if the plasma is non-dissipative, has a flow velocity parallel to the external enveloping field and is exposed to radiation from a time-independent magnetic dipole then, along the plane plasma–magnetic field interface in the steady state, either (i) a single stationary wave may be formed upstream, if the flow is subsonic, or (ii) an infinity of stationary waves are superposed downstream, if the flow is supersonic.

In the problems investigated by Kruskal & Schwarzschild, Tayler and Savage, the plasma model is represented by an electrically conducting fluid continuum conforming to purely hydromagnetic equations.

1.1. *Configuration of motion*

To follow up this line of study, let us consider a stable configuration in equilibrium comprising an infinite-length cylindrical jet of non-gravitating, perfectly conducting, inviscid, compressible fluid permeated internally by a magnetic field and confined externally by another magnetic field occupying an infinite vacuous space. Both magnetic fields are aligned with the axial jet stream, the interior field being permanently frozen and trapped within the jet column, which, likewise, freezes the exterior field upon its otherwise free-surface. Small axisymmetric excitations are initiated by a stationary, weak, azimuthal current arbitrarily distributed over some finite-length tubular conductor co-axially positioned in the vacuum field. As in normal practice, a non-relativistic version is proposed.

1.2. *An outline*

Equations are originally formulated for unsteady motion to allow the application (in accordance with Lighthill (1960)) of a radiation condition that will ensure the uniqueness of any particular solution in the ultimate steady state, attributed to a steady source current. A general asymptotic solution, incorporating a radiation condition and valid at large axial distances from the source, is established for various ranges of the flow parameters in which the dispersion relation possesses neither a vanishing root nor a repeated real root. This solution is either negligible (in certain trivial cases) or is, essentially, a discrete superposition of stationary wave functions over the set of real, non-vanishing, distinct roots of the dispersion relation. It then transpires that, as a consequence of the imposed radiation condition, each distinct wave function exists only on one side (viz. upstream or downstream) of the generating source, justifiably, the side into which the sustaining energy is transported. There is no wave attenuation in the axial direction, but, within the vacuum field, radial attenuation occurs with a transverse withdrawl from the axis of symmetry.

The type of steady-state results obtained depends on one of the following criteria:

$$M^2+A^2 < 1, \tag{E 1}$$

$$M > 1, \quad A < 1, \tag{E 2}$$

$$M < 1, \quad A > 1 \begin{cases} \lambda\beta \geqslant 1, & \text{(E 3a)} \\ \lambda\beta < 1, & \text{(E 3b)} \end{cases}$$

$$M > 1, \quad A > 1, \tag{H 1}$$

$$M < 1, \quad A < 1, \quad M^2+A^2 > 1, \tag{H 2}$$

where M and A are, respectively, the Mach and Alfvén numbers of the jet stream, and $\lambda\beta$ is a known parametric function of M and A as well as A_0 (the ratio of the flow speed of the jet to the Alfvén speed involving the external confining field, i.e. an Alfvén number for the fluid–vacuum interface). The flow regimes (E 1)–(E 3) (or (H 1) and (H 2)) correspond to, adopting the terminologies of McCune, Resler and Sears (McCune & Resler 1960; Sears & Resler 1964), an elliptic (or hyperbolic) motion of the jet column. Under any one of the conditions (E 1), (E 2) or (E 3*a*) perturbations are asymptotically negligible. If (E 3*b*) holds, then our general superposed solution reduces to a single term, precisely, a single stationary wave. Moreover, for the subregime $M < 1$, $M^2+A^2 \geqslant 2$ of (E 3), we can prove that this particular wave must appear upstream. For both hyperbolic flow regimes (H 1) and (H 2), the superposition is one consisting of an infinite number of stationary waves, all of which are, in the case of (H 1), to be found downstream. [Note that if the jet is free from an internally trapped field then $A = \infty$, in which case, of the five jet regimes, only (E 3) and (H 1), together with their associated results, are relevant and correspond to $M < 1$ and $M > 1$, respectively. Thus physical conclusions agree with those of Savage for his rectangular (i.e. Cartesian) analogue of the cylindrical field-free jet.]

2. Equations of motion

With reference to a cylindrical (x, r, θ) co-ordinate frame, the x axis is chosen along the axis of symmetry of the infinite-length equilibrial jet column $r \leqslant r_0$ (see §1.1). Present in the surrounding vacuum $r > r_0$ is an azimuthal current source of density

$$\mathbf{J}(x, r, t) = (0, 0, J(x, r, t)), \tag{2.1}$$

being carried along a co-axial tubular conductor of finite length $2l$:

$$J \equiv 0 \quad \text{for} \quad |x| > l, \tag{2.2}$$

and bounded by the profiles $r = r_1(x)$ $(> r_0)$ and $r = r_2(x)$ $(> r_1(x))$:

$$J \equiv 0 \quad \text{in} \quad r_0 \leqslant r < r_1(x) \qquad \text{and} \qquad r_2(x) < r < \infty. \tag{2.3}$$

The current distribution $J(x, r, t)$ is clearly axisymmetric and is assumed to be small in magnitude. Excitations created by it are likewise axisymmetric and correspondingly 'weak'. In this paper, we restrict ourselves to a non-relativistic treatment, wherein the speed of light infinitely exceeds all other characteristic speeds of propagation, namely, the speeds of sound and Alfvén waves. Equations formulated are in electromagnetic units and linearized (wherever nonlinear).

In the external vacuum, whose magnetic permeability is unity, the perturbation $\mathbf{H}$ (induced by the current source) from the uniform confining axial magnetic field $\mathbf{H}_0 = (H_0, 0, 0)$ satisfies

$$\operatorname{curl} \mathbf{H} = 4\pi \mathbf{J}(x, r, t), \tag{2.4}$$

$$\operatorname{div} \mathbf{H} = 0. \tag{2.5}$$

[Note that the displacement current term normally associated, in electromagnetic theory, with (2.4) is virtually non-existent under a non-relativistic hypothesis.] From (2.1) and (2.4), it follows that $\mathbf{H}$ possesses only axial and radial components:

$$\mathbf{H} = (H_x, H_r, 0). \tag{2.6}$$

Suppose that a magnetic line of force in the total external field $\mathbf{H}+\mathbf{H}_0$ is radially displaced by the amount ξ. Then

$$H_0\,\partial\xi/\partial x = H_r. \tag{2.7}$$

Also, a boundedness condition is required, viz.

$$|\xi| < \infty \quad \text{as} \quad r \to \infty. \tag{2.8}$$

We next turn our attention to the disturbance field transmitted to the jet column of inviscid compressible fluid having magnetic permeability μ, say, and infinite electrical conductivity. Let us first describe the initial uniform state of the fluid by its density ρ and pressure Π, the speed c of sound (in the fluid), the jet flow velocity $(U, 0, 0)$ and the magnetic field $(H, 0, 0)$ trapped within it. Variations in the flow and magnetic fields are defined by $\mathbf{u}$ and $\mathbf{h}$, whose (x, r) components are, respectively, (u, v) and (h, g). Finally, we let p denote the fluid pressure perturbation from Π. Whereupon, the motion within (approximately) $r \leqslant r_0$ is governed by the following equations:

$$\frac{\partial p}{\partial t}+U\frac{\partial p}{\partial x}+c^2\rho\operatorname{div}\mathbf{u} = 0, \tag{2.9}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+U\frac{\partial \mathbf{u}}{\partial x}\right)-\frac{\mu H}{4\pi}\frac{\partial \mathbf{h}}{\partial x}+\operatorname{grad}\left(p+\frac{\mu H}{4\pi}h\right) = \mathbf{0}, \tag{2.10}$$

$$\operatorname{div}\mathbf{h} = 0, \tag{2.11}$$

$$\frac{\partial \mathbf{h}}{\partial t}+U\frac{\partial \mathbf{h}}{\partial x}-H\frac{\partial \mathbf{u}}{\partial x}+(H,0,0)\operatorname{div}\mathbf{u} = \mathbf{0}, \tag{2.12}$$

and the symmetry condition

$$v = g = 0 \quad \text{at} \quad r = 0. \tag{2.13}$$

Also, if η is the radial elevation of a magnetic line of the total internal magnetic field, then

$$H\,\partial\eta/\partial x = g. \tag{2.14}$$

Hence the r component of (2.12) gives

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}+U\frac{\partial\eta}{\partial x}-v\right) = 0. \tag{2.15}$$

We now stipulate conditions at the interface (originally coinciding with $r = r_0$). First, we note that (2.15) implies that $\eta|_{r=r_0}$ is, in fact, a deformation of the interface, i.e. the jet profile. Across this profile, the normal component of magnetic field must be continuous, thus requiring that

$$\xi = \eta \quad \text{at} \quad r = r_0. \tag{2.16}$$

A second condition is that the internal magnetic pressure plus fluid pressure must balance the external magnetic pressure at the interface. This condition is reducible to

$$p+\frac{\mu H}{4\pi}h = \frac{H_0}{4\pi}H_x \quad \text{at} \quad r = r_0, \tag{2.17}$$

because in the undisturbed equilibrium

$$\Pi+\mu H^2/8\pi = H_0^2/8\pi. \tag{2.18}$$

3. A Fourier analysis

In this paper, we are mainly interested in deriving a (possible) steady-state wave solution which is unique. With this aim in mind, we let the strength of the current source (2.1) accumulate exponentially with time:

$$-(4\pi/H_0)\,J(x,r,t) = e^{\epsilon t}\,\chi(x,r) \quad (\epsilon > 0). \tag{3.1}$$

Introducing Fourier transforms, each indicated by an asterisk, we define

$$\chi^*(\alpha,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(x,r)\,e^{-i\alpha x}\,dx, \tag{3.2}$$

whose inverse is

$$\chi(x,r) = \int_{-\infty}^{\infty} \chi^*(\alpha,r)\,e^{i\alpha x}\,d\alpha. \tag{3.3}$$

For a wave solution in the steady state to be unique (i.e. unaccompanied by arbitrary complementary functions), it must satisfy a radiation condition. This condition can be incorporated in the manner proposed by Lighthill (1960), namely, by initially allowing each perturbation function to develop exponentially with time in step with the 'growing' source:

$$\xi = e^{\epsilon t}\int_{-\infty}^{\infty} \xi^*(\alpha,r;\epsilon)\,e^{i\alpha x}\,d\alpha. \tag{3.4}$$

Since any free perturbation (which corresponds to a complementary function) encountered originates independently of the source, and so maintains a time-independent amplitude, its effect at large t (i.e. near the steady state) is negligible compared with that of the forced perturbation determined to a magnitude of order $e^{\epsilon t}$. Thus, a steady-state solution for ξ, say, corresponding to the forcing current specified by (2.1)–(2.3) with

$$-(4\pi/H_0)\,J = \chi(x,r),$$

can be uniquely constructed by evaluating its integral representation (3.4) for small positive ϵ and then letting ϵ tend to zero.

3.1. *The region $r > r_0$*

Fourier transformation (using (3.1)–(3.4)) of (2.4)–(2.7) results in

$$\partial H_x^*/\partial r - i\alpha H_r^* = H_0\chi^*(\alpha,r), \tag{3.5}$$

$$\frac{\partial H_r^*}{\partial r}+\frac{H_r^*}{r}+i\alpha H_x^* = 0, \tag{3.6}$$

$$i\alpha H_0\xi^* = H_r^*, \tag{3.7}$$

these three equations being reducible to

$$\frac{\partial^2\xi^*}{\partial r^2}+\frac{1}{r}\frac{\partial\xi^*}{\partial r}-\left(\alpha^2+\frac{1}{r^2}\right)\xi^* = -\chi^*(\alpha,r), \tag{3.8}$$

an inhomogeneous Bessel equation for the quantity ξ^*, which, in view of (2.8), also satisfies the condition

$$|\xi^*| < \infty \quad \text{as} \quad r \to \infty. \tag{3.9}$$

After solving (3.8) by the method of variation of parameters, incorporating (3.9) and using the appropriate recurrence and Wronskian relations of the modified Bessel functions $I_n(z)$ and $K_n(z)$ (Watson 1944, 3.71), we arrive at

$$\xi^* = K_1(r|\alpha|)\left(C+\int_{r_0}^{r} I_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa\right)+I_1(r|\alpha|)\int_{r}^{R} K_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa, \tag{3.10}$$

$C = C(\alpha, \epsilon)$ being (for the moment) arbitrary and the positive constant $R \geqslant r_2(x)$ so that $\chi \equiv 0$ in $r > R$ (cf. (2.3)).

3.2. *The region* $r \leqslant r_0$

Likewise, on applying the Fourier integral (3.4) to (2.9)–(2.15) and retaining relevant components of vector equations, we have

$$(i\alpha U+\epsilon)\,p^*+c^2\rho\left(\frac{\partial v^*}{\partial r}+\frac{v^*}{r}+i\alpha u^*\right) = 0, \tag{3.11}$$

$$\rho(i\alpha U+\epsilon)\,u^*-i\alpha\frac{\mu H}{4\pi}h^*+i\alpha\left(p^*+\frac{\mu H}{4\pi}h^*\right) = 0, \tag{3.12}$$

$$\rho(i\alpha U+\epsilon)\,v^*-i\alpha\frac{\mu H}{4\pi}g^*+\frac{\partial}{\partial r}\left(p^*+\frac{\mu H}{4\pi}h^*\right) = 0, \tag{3.13}$$

$$\frac{\partial g^*}{\partial r}+\frac{g^*}{r}+i\alpha h^* = 0, \tag{3.14}$$

$$(i\alpha U+\epsilon)\,h^*-i\alpha H u^*+H\left(\frac{\partial v^*}{\partial r}+\frac{v^*}{r}+i\alpha u^*\right) = 0, \tag{3.15}$$

$$v^* = g^* = 0 \quad \text{at} \quad r = 0, \tag{3.16}$$

$$i\alpha H\eta^* = g^*, \tag{3.17}$$

$$(i\alpha U+\epsilon)\,\eta^* = v^*. \tag{3.18}$$

Suppose that the flow parameter β is defined by

$$\beta \equiv \beta(U) = \left(\frac{(A^2-1)(1-M^2)}{M^2+A^2-1}\right)^{\frac{1}{2}}, \tag{3.19}$$

where $a = (\mu H^2/4\pi\rho)^{\frac{1}{2}}$ is the (internal) Alfvén speed of the fluid column, and $M = U/c$ and $A = U/a$ are, respectively, its Mach number and Alfvén number. For the compressible jet, we assume that $M \neq 1$, $A \neq 1$, $M^2+A^2 \neq 1$ and $U > 0$. Furthermore, let us define

$$\beta_\epsilon = \beta(U-i\epsilon\alpha^{-1}). \tag{3.20}$$

We can now reduce (3.11), (3.12) and (3.15) to

$$\alpha^2(1-\beta_\epsilon^2)\left(p^*+\frac{\mu H}{4\pi}h^*\right) = \rho(i\alpha U+\epsilon)\left(\frac{\partial v^*}{\partial r}+\frac{v^*}{r}+i\alpha u^*\right),$$

and also manipulate (3.12)–(3.14) to obtain

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\alpha^2\right)\left(p^*+\frac{\mu H}{4\pi}h^*\right) = -\rho(i\alpha U+\epsilon)\left(\frac{\partial v^*}{\partial r}+\frac{v^*}{r}+i\alpha u^*\right).$$

By adding, we obtain Bessel's equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\alpha^2\beta_\epsilon^2\right)\left(p^*+\frac{\mu H}{4\pi}h^*\right) = 0. \tag{3.21}$$

Appropriate boundary conditions are, by (3.13) and (3.16),

$$\frac{\partial}{\partial r}\left(p^*+\frac{\mu H}{4\pi}h^*\right) = 0 \quad \text{at} \quad r = 0, \tag{3.22}$$

and, in view of (2.17), (3.6) and (3.7),

$$p^*+\frac{\mu H}{4\pi}h^* = -\frac{H_0^2}{4\pi}\left(\frac{\partial \xi^*}{\partial r}+\frac{\xi^*}{r}\right) \quad \text{at} \quad r = r_0. \tag{3.23}$$

The solution to (3.21)–(3.23) within $r \leqslant r_0$ is then

$$p^*+\frac{\mu H}{4\pi}h^* = -\frac{H_0^2}{4\pi}\frac{I_0(\beta_\epsilon r|\alpha|)}{I_0(\beta_\epsilon r_0|\alpha|)}\left(\frac{\partial \xi^*}{\partial r}+\frac{\xi^*}{r}\right)_{r=r_0}. \tag{3.24}$$

Let $a_0 = (H_0^2/4\pi\rho)^{\frac{1}{2}}$, the (interface) Alfvén speed arising from the contact between the fluid column and its confining magnetic field. Substitution of (3.17), (3·18) and (3.24) into (3.13) then yields

$$\eta^* = \frac{\beta_\epsilon a_0^2}{a^2-(U-i\epsilon\alpha^{-1})^2}\frac{I_1(\beta_\epsilon r|\alpha|)}{|\alpha|\,I_0(\beta_\epsilon r_0|\alpha|)}\left(\frac{\partial \xi^*}{\partial r}+\frac{\xi^*}{r}\right)_{r=r_0}, \tag{3.25}$$

which together with the Fourier transform of (2.16), i.e.

$$\eta^* = \xi^* \quad \text{at} \quad r = r_0, \tag{3.26}$$

implies

$$\eta^* = \frac{I_1(\beta_\epsilon r|\alpha|)}{I_1(\beta_\epsilon r_0|\alpha|)}(\xi^*)_{r=r_0}. \tag{3.27}$$

Either of the expressions (3.25) or (3.27) for η^* is valid in $r \leqslant r_0$.

3.3. *The unique solutions for ξ^* and η^**

Let us introduce another parameter of motion, namely,

$$\lambda \equiv \lambda(U) = \frac{a^2}{a_0^2}\frac{A^2-1}{\beta^2}, \tag{3.28}$$

and define

$$\lambda_\epsilon = \lambda(U-i\epsilon\alpha^{-1}). \tag{3.29}$$

From (3.25) and (3.26), we then have

$$r\frac{\partial \xi^*}{\partial r}+\xi^*+\lambda_\epsilon\beta_\epsilon r_0|\alpha|\frac{I_0(\beta_\epsilon r_0|\alpha|)}{I_1(\beta_\epsilon r_0|\alpha|)}\xi^* = 0 \quad \text{at} \quad r = r_0, \tag{3.30}$$

the boundary condition which, when applied to (3.10), would complete the evaluation of a unique ξ^*. Thus, in (3.10), C is determined, on employing the Bessel recurrence relations, by

$$C = \frac{F(\alpha, U-i\epsilon\alpha^{-1})}{G(\alpha, U-i\epsilon\alpha^{-1})}\int_{r_0}^{R} K_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa, \tag{3.31}$$

in which

$$F(\alpha, U-i\epsilon\alpha^{-1}) = I_0(r_0|\alpha|)\,I_1(\beta_\epsilon r_0|\alpha|)+\lambda_\epsilon\beta_\epsilon I_1(r_0|\alpha|)\,I_0(\beta_\epsilon r_0|\alpha|), \tag{3.32}$$

$$G(\alpha, U-i\epsilon\alpha^{-1}) = K_0(r_0|\alpha|)\,I_1(\beta_\epsilon r_0|\alpha|)-\lambda_\epsilon\beta_\epsilon K_1(r_0|\alpha|)\,I_0(\beta_\epsilon r_0|\alpha|). \tag{3.33}$$

Thus, we eventually arrive at the expression

$$\begin{aligned}\xi^* = {} & \frac{F(\alpha, U-i\epsilon\alpha^{-1})}{G(\alpha, U-i\epsilon\alpha^{-1})}K_1(r|\alpha|)\int_{r_0}^{R}K_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa \\ & +K_1(r|\alpha|)\int_{r_0}^{r}I_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa+I_1(r|\alpha|)\int_{r}^{R}K_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa \quad (r \geqslant r_0).\end{aligned} \tag{3.34}$$

Its application to (3.27), incorporating the Wronskian relation, leads to

$$\eta^* = \frac{I_1(\beta_\epsilon r|\alpha|)}{r_0|\alpha|\,G(\alpha, U-i\epsilon\alpha^{-1})}\int_{r_0}^{R}K_1(\kappa|\alpha|)\,\chi^*(\alpha,\kappa)\,\kappa\,d\kappa \quad (r \leqslant r_0). \tag{3.35}$$

3.4. *Steady-state integrals*

The Fourier inversion (using limiting form of (3.4), as $\epsilon \to 0_+$) of (3.35), taking into account (3.2), (2.2) and (2.3), yields for the steady-state motion

$$\eta(x,r) = \int_{-l}^{l}dz\int_{r_1(z)}^{r_2(z)}\chi(z,\kappa)\,\kappa\,d\kappa\,\lim_{\epsilon\to 0_+}\frac{1}{2\pi r_0}\int_{-\infty}^{\infty}\frac{I_1(\beta_\epsilon r|\alpha|)\,K_1(\kappa|\alpha|)}{|\alpha|\,G(\alpha, U-i\epsilon\alpha^{-1})}e^{i\alpha(x-z)}\,d\alpha. \tag{3.36}$$

Likewise, if
$$\xi_1(x,r) = \int_{-l}^{l}dz\int_{r_1(z)}^{r_2(z)}\Phi(x,r;z,\kappa)\,\chi(z,\kappa)\,\kappa\,d\kappa, \tag{3.37}$$
where

$$\Phi(x,r;z,\kappa) = \lim_{\epsilon\to 0_+}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\alpha, U-i\epsilon\alpha^{-1})}{G(\alpha, U-i\epsilon\alpha^{-1})}K_1(r|\alpha|)\,K_1(\kappa|\alpha|)\,e^{i\alpha(x-z)}\,d\alpha, \tag{3.38}$$

then a similar inversion of (3.34) shows that, in the steady state, we may write

$$\xi(x,r) = \xi_1(x,r)+\xi_2(x,r). \tag{3.39}$$

The term $\xi_2(x,r)$ occurring in (3.39) can be expressed in the form

$$\begin{aligned}\xi_2(x,r) = {} & \frac{1}{\pi}\int_{-l}^{l}dz\int_{r_1(z)}^{r}\chi(z,\kappa)\,\kappa\,d\kappa\int_0^\infty K_1(r\alpha)\,I_1(\kappa\alpha)\cos[\alpha(x-z)]\,d\alpha \\ & +\frac{1}{\pi}\int_{-l}^{l}dz\int_{r}^{r_2(z)}\chi(z,\kappa)\,\kappa\,d\kappa\int_0^\infty K_1(\kappa\alpha)\,I_1(r\alpha)\cos[\alpha(x-z)]\,d\alpha,\end{aligned}$$

which is reducible by a cosine integral formula (see Erdélyi *et al.* 1954, 1.12) to

$$\xi_2(x,r) = \frac{1}{2\pi}\int_{-l}^{l}dz\int_{r_1(z)}^{r_2(z)}\left(\frac{\kappa}{r}\right)^{\frac{1}{2}}\chi(z,\kappa)\,Q_{\frac{1}{2}}\left[\frac{\kappa^2+(x-z)^2+r^2}{2\kappa r}\right]d\kappa. \tag{3.40}$$

Here, $Q_{\frac{1}{2}}(z)$ is a Legendre function of the second kind and is related to a hypergeometric function by

$$\begin{aligned}Q_{\frac{1}{2}}(z) &= \pi 2^{-\frac{5}{2}}z^{-\frac{3}{2}}\,{}_2F_1(\tfrac{3}{4},\tfrac{5}{4};2;1/z^2), \\ &\sim \pi 2^{-\frac{5}{2}}z^{-\frac{3}{2}} \quad \text{as} \quad |z| \to \infty.\end{aligned}$$

Thus, in particular,
$$\xi_2(x,r) = O(x^{-3}) \quad \text{for} \quad |x| \gg l. \tag{3.41}$$

Now, the expression (3.40) does not, in any way, involve the radius r_0 of the fluid column. Hence the value of $\xi_2(x,r)$ is independent of conditions at the interface $r = r_0$ and must, therefore, represent the radiation received directly from the current source without having undergone any reflexion at the interface, that is, an incident wave.

It follows that the component $\xi_1(x,r)$ is a measure of the radiation reflected from the interface. The resultant effect (of incidence plus reflexion) associated with $\xi(x,r)$ is transmitted across the interface and propagated into the jet column by the perturbation $\eta(x,r)$.

In the case of a uniform source current confined to a circular filament positioned at $x = z$, $r = \kappa$, so that

$$\chi(x,r) = r^{-1}\delta(r-\kappa)\,\delta(x-z), \tag{3.42}$$

$\delta(x)$ being the Dirac delta function, (3.41) becomes

$$\xi_1(x,r) = \Phi(x,r;z,\kappa).$$

Obviously, $\Phi(x,r;z,\kappa)$ is a Green's function of the problem.

4. An asymptotic approximation

We shall now seek an asymptotic approximation to the Green's function $\Phi(x,r;z,\kappa)$ at large axial distances from the current loop (3.42) at $x = z$. The starting point is (3.38), which can be written in the form

$$\Phi(x,r;z,\kappa) = \lim_{\epsilon\to 0+}\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\alpha,\epsilon)\,e^{i\alpha(x-z)}\,d\alpha, \tag{4.1}$$

where
$$\phi(\alpha,\epsilon) = \frac{\alpha F(\alpha,U)-i\epsilon F_U(\alpha,U)}{\alpha G(\alpha,U)-i\epsilon G_U(\alpha,U)}K_1(r|\alpha|)\,K_1(\kappa|\alpha|), \tag{4.2}$$

with G_U denoting $\partial G/\partial U$ and (cf. (3.32) and (3.33))

$$F(\alpha,U) = I_0(r_0|\alpha|)\,I_1(\beta r_0|\alpha|)+\lambda\beta I_1(r_0|\alpha|)\,I_0(\beta r_0|\alpha|), \tag{4.3}$$

$$G(\alpha,U) = K_0(r_0|\alpha|)\,I_1(\beta r_0|\alpha|)-\lambda\beta K_1(r_0|\alpha|)\,I_0(\beta r_0|\alpha|). \tag{4.4}$$

The result achieved will then be used to generate a corresponding asymptotic estimation of $\xi(x,r)$.

We use contour integration to approximate to (4.1) in two parts, viz.

$$\int_0^{\infty} \quad\text{and}\quad \int_{-\infty}^{0}.$$

The integral from nought to infinity is evaluated with the variable $|\alpha|$ replaced by α, and the real path $(0,\infty)$ suitably deformed into the complex-α plane. Likewise, the integral from minus infinity to zero is evaluated with $|\alpha|$ replaced by $-\alpha$, and the real path $(-\infty,0)$ similarly deformed. In both cases, the contour deformation must take into account the logarithmic branch point of the functions K_0 and K_1 at $\alpha = 0$. Otherwise, for complex $\alpha \neq 0$, the integrand factor $\phi(\alpha,\epsilon)$ is meromorphic, its only singularities being poles. These derive from the zeros of $G(\alpha,U)$, for example, when sufficiently near a real simple zero $\alpha = \alpha_\nu$ of $G(\alpha, U)$, the denominator in (4.2) may be approximated as $\epsilon \to 0_+$ by (cf. Lighthill 1960)

$$\alpha_\nu G_\alpha(\alpha_\nu, U)\,[\alpha-(\alpha_\nu+i\epsilon/V(\alpha_\nu))], \tag{4.5}$$

in which G_α denotes $\partial G/\partial\alpha$ and

$$V(\alpha) = \alpha G_\alpha(\alpha, U)/G_U(\alpha, U); \tag{4.6}$$

the implication of (4.5) is that $\phi(\alpha.\epsilon)$ has, as $\epsilon \to 0_+$, a simple pole at

$$\alpha = \alpha_\nu + i\epsilon/V(\alpha_\nu), \tag{4.7}$$

situated at a small vertical distance from the position $\alpha = \alpha_\nu$ on the $\operatorname{Re}(\alpha)$ axis. From here on, we shall work under the hypothesis that the initial undisturbed configuration of motion is such that all (possible) real zeros of $G(\alpha, U)$ are simple (i.e. distinct) and non-vanishing. The corresponding poles of $\phi(\alpha, \epsilon)$ are therefore simple and distributed in accordance with (4.7), well outside a neighbourhood of the branch point at $\alpha = 0$. [Note that an initial undisturbed configuration, which is responsible for $G(\alpha, U)$ acquiring a real (repeated) zero of order m ($\geqslant 2$), say, must be avoided as this will impart to each perturbation solution a term which is of $O(x^{m-1})$ as $|x| \to \infty$ and hence, clearly, incompatible with linearized equations. Moreover, in such a system, it can be shown that the originally unsteady perturbed motion can never develop a steady state.]

Since the poles of a meromorphic function are isolated, there is a narrow strip $|\operatorname{Im}(\alpha)| \leqslant k$, say, about the $\operatorname{Re}(\alpha)$ axis within which $\phi(\alpha, \epsilon)$ is analytic except for its branch point at $\alpha = 0$ and (possible) poles which are slightly displaced, as indicated by (4.7), from the real positions $\alpha = \alpha_\nu$. The real integral path $(0, \infty)$ for Φ may therefore be deformed, as illustrated in figure 1, into the straight segment proceeding from 0 to ik and thence to $ik+\infty$ whenever $x > z$ (or into the straight segment proceeding from 0 to $-ik$ and thence to $-ik+\infty$ whenever $x < z$). A similar deformation for the path $(-\infty, 0)$ is also carried out (see figure 1). The only poles crossed by each such contour deformation are the ones associated, via (4.7), with those real simple zeros α_ν which necessarily satisfy $V(\alpha_\nu) > 0$ (or $V(\alpha_\nu) < 0$). Whence, by the residue theory, the contribution

$$\pm i \sum_\nu \lim_{\epsilon\to 0_+} \underset{\alpha_\nu+i\epsilon/V(\alpha_\nu)}{\text{residue}}\ \phi(\alpha, \epsilon)\, e^{i\alpha(x-z)} \tag{4.8}$$

to Φ in the region $x \gtrless z$ results (according as $V(\alpha_\nu) \gtrless 0$). Along each of the vertical branch cuts about the $\operatorname{Im}(\alpha)$ axis, the diverted contour integral, estimated as a Laplace integral (see, for example, Erdélyi 1956), is of $O(|x-z|^{-1})$ as $|x-z| \to \infty$. This is, however, dominated by the residue sum (4.8). Also, from the divided horizontal segment $\operatorname{Im}\alpha = k$ (or $-k$), there is a contribution of a bounded Fourier integral. This is of $O(e^{-k|x-z|})$, which is evidently negligible. Whereupon, we finally arrive at the conclusion that, in general,

$$\Phi(x, r; z, \kappa) \sim i \sum_\nu \frac{F(\alpha_\nu, U)}{G_\alpha(\alpha_\nu, U)} K_1(r|\alpha_\nu|)\, K_1(\kappa|\alpha_\nu|)\, e^{i\alpha_\nu(x-z)} \operatorname{sgn}[V(\alpha_\nu)]\, H[(x-z)\, V(\alpha_\nu)] \tag{4.9}$$

as $|x-z| \to \infty$. Here $H(x)$ denotes the Heaviside unit step function.

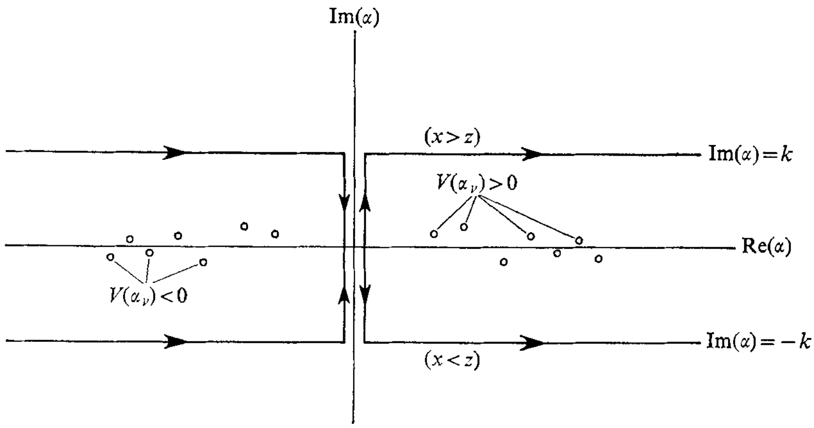

FIGURE 1. The deformed contours enclosing simple poles slightly displaced from real positions at $\alpha = \alpha_\nu$.

Since (4.4) implies that, in particular, $G(\alpha, U) \equiv G(|\alpha|, U)$ when α is real, the real zeros of $G(\alpha, U)$ must actually appear in symmetric pairs at $\alpha = \pm |\alpha_\nu|$. Also, we note that, for real $\alpha \neq 0$, $G_\alpha(\alpha, U) \equiv (\operatorname{sgn}\alpha)\, G_{|\alpha|}(|\alpha|, U)$ and, in view of (4.6),

$$V(\alpha) \equiv V(|\alpha|) = |\alpha|\, G_{|\alpha|}(|\alpha|, U)/G_U(|\alpha|, U), \tag{4.10}$$

where $G_{|\alpha|} = \partial G/\partial|\alpha|$. Furthermore, it follows from (4.3), (4.4) and the Wronskian relation that

$$F(\alpha_\nu, U) = I_1(\beta r_0|\alpha_\nu|)/r_0|\alpha_\nu|\, K_1(r_0|\alpha_\nu|). \tag{4.11}$$

We can now simplify the form (4.9), substitute it into (3.37) for sufficiently large $|x|$ and, finally, employ (3.39) and (3.41). A similar analysis is applicable to (3.36). [Note that with regard to the inner α-integral of (3.36), there is no pole associated with the origin $\alpha = 0$ since at this point $I_1(\beta r|\alpha|)$ vanishes.] Thus, whenever $|x| \gg l$

$$\xi(x,r) \sim \sum_\nu \frac{I_1(\beta r_0|\alpha_\nu|)\, K_1(r|\alpha_\nu|)\, \chi_\nu(x)}{G_{|\alpha|}(|\alpha_\nu|, U)\, K_1(r_0|\alpha_\nu|)\, r_0|\alpha_\nu|} \operatorname{sgn}[V(|\alpha_\nu|)]\, H[xV(|\alpha_\nu|)] \quad (r \geqslant r_0), \tag{4.12}$$

$$\eta(x,r) \sim \sum_\nu \frac{I_1(\beta r|\alpha_\nu|)\, \chi_\nu(x)}{G_{|\alpha|}(|\alpha_\nu|, U)\, r_0|\alpha_\nu|} \operatorname{sgn}[V(|\alpha_\nu|)]\, H[xV(|\alpha_\nu|)] \quad (r \leqslant r_0), \tag{4.13}$$

where
$$\chi_\nu(x) = 2\int_{-l}^{l} \sin[|\alpha_\nu|\,(z-x)]\, dz \int_{r_1(z)}^{r_2(z)} K_1(\kappa|\alpha_\nu|)\, \chi(z,\kappa)\, \kappa\, d\kappa \tag{4.14}$$

and $\sum_\nu$ denotes a summation ranging over all (possible) real positive roots $\alpha = |\alpha_\nu|$ (these being, necessarily, distinct) of

$$G(\alpha, U) = 0, \tag{4.15}$$

with $G(\alpha, U)$ determined by (4.4).

4.1. *A physical interpretation*

The results (4.12) and (4.13) reveal that any substantial perturbation motion, generated within $|x| \gg l$, occurs as a superposed collection of stationary wave functions, each characterized by its wavenumber $|\alpha_\nu|$. There is no dissipation in the axial direction. In the external domain $r \geqslant r_0$, however, wave amplitudes are radially attenuated as $r \to \infty$ (via the asymptotic formula for $K_n(z)$) like $e^{-r|\alpha_\nu|}(r|\alpha_\nu|)^{-\frac{1}{2}}$.

Ostensibly, the wave with wavenumber $|\alpha_\nu|$ emerges in the downstream region $x > 0$ (or upstream region $x < 0$) if and only if $V(|\alpha_\nu|) > 0$ (or < 0). This mathematical phenomenon can be accorded a physical significance as follows. First, we deduce from (3.4) and (3.34) that the travelling wave

$$\xi = \xi^*(\alpha, r, -i\omega)\, e^{i(\alpha x - \omega t)} \tag{4.16}$$

constitutes a non-trivial complementary function (i.e. a solution to the associated source-free motion in which $\chi \equiv 0$) if, since

$$G(\alpha, U - \omega\alpha^{-1})\, \xi^* = 0,$$

the phase velocity $q = \omega\alpha^{-1}$ satisfies the dispersion relation

$$G(\alpha, U - q) = 0, \tag{4.17}$$

this being a necessary condition. Hence

$$\frac{\partial q}{\partial \alpha} = \frac{G_\alpha(\alpha, U-q)}{G_{U-q}(\alpha, U-q)},$$

with $G\alpha = \partial G/\partial\alpha$ and $G_{U-q} = \partial G/\partial(U-q)$. Hence, the group velocity $\partial\omega/\partial\alpha$ (or velocity of energy propagation) in the positive x direction is determined by

$$\frac{\partial \omega}{\partial \alpha} - q = \frac{\alpha G_\alpha(\alpha, U-q)}{G_{U-q}(\alpha, U-q)}. \tag{4.18}$$

Now, let $\omega = 0$. Then $q = 0$, in which case ξ possesses a stationary wave representation. By virtue of (4.6) and (4.18), the group velocity of this stationary wave system is therefore

$$\partial\omega/\partial\alpha = V(\alpha), \tag{4.19}$$

where α is governed by the reduced dispersion relation (4.15). Thus a stationary wave function appears downstream (or upstream) if and only if its associated energy is propagated downstream (or upstream) from the radiating source. This observation is consistent with an interpretation of Lighthill (1960, 1965) for the radiation condition.

5. Physical results

In view of (4.4), the positive simple zeros $\alpha = |\alpha_\nu|$ of $G(\alpha, U)$ are related by $r_0|\alpha_\nu| = z_\nu$ to the (positive) distinct roots $z = z_\nu$ of the equation

$$Z_1(z) = \lambda\beta Z_2(\beta z) \quad (z > 0), \tag{5.1}$$

where the functions $Z_1(z)$ and $Z_2(z)$ are given by

$$Z_1(z) = \frac{K_0(z)}{K_1(z)}, \quad Z_2(z) = \frac{I_0(z)}{I_1(z)} \tag{5.2}$$

and have the first derivatives

$$Z_1'(z) = Z_1(z)\left(\frac{1}{z}-\frac{1}{F_1(z)}\right), \tag{5.3}$$

$$Z_2'(z) = Z_2(z)\left(\frac{1}{z}-\frac{1}{F_2(z)}\right), \tag{5.4}$$

in which

$$F_1(z) = \frac{K_0(z)\,K_1(z)}{K_1^2(z)-K_0^2(z)}, \tag{5.5}$$

$$F_2(z) = \frac{I_0(z)\,I_1(z)}{I_0^2(z)-I_1^2(z)}. \tag{5.6}$$

Whence, from (4.4) and (4.10), the quantities $G_{|\alpha|}(|\alpha_\nu|, U)$ and $V(|\alpha_\nu|)$, required in the estimations for $\xi(x,r)$ and $\eta(x,r)$, can now be explicitly formulated. Thus,

$$G_{|\alpha|}(|\alpha_\nu|, U) = r_0\lambda\beta K_1(r_0|\alpha_\nu|)\,I_0(\beta r_0|\alpha_\nu|)\left[\frac{\beta}{F_2(\beta r_0|\alpha_\nu|)}-\frac{1}{F_1(r_0|\alpha_\nu|)}\right], \tag{5.7}$$

while

$$V(|\alpha_\nu|) = \frac{U[1/F_1(r_0|\alpha_\nu|)-\beta/F_2(\beta r_0|\alpha_\nu|)]}{2A^2/r_0|\alpha_\nu|\,(A^2-1)-\tfrac{1}{2}U(\partial\beta^2/\partial U)/\beta F_2(\beta r_0|\alpha_\nu|)}, \tag{5.8}$$

the group velocity at wavenumber $|\alpha_\nu|$. With reference to (5.8),

$$-\frac{U}{2}\frac{\partial\beta^2}{\partial U} = \frac{M^2A^2(M^2+A^2-2)}{(M^2+A^2-1)^2}. \tag{5.9}$$

The motion can be classified into two main categories, corresponding to $\beta^2 > 0$ and $\beta^2 < 0$. We note (cf. McCune & Resler 1960; Sears & Resler 1964) that $\beta^2 > 0$ (or < 0) is, in fact, the condition that the spatial differential operator (in $\partial/\partial x$, $\partial/\partial r$) associated with (3.21) is elliptic (or hyperbolic), in which case, the jet flow may be referred to as elliptic (or hyperbolic). The perturbation motion of the exterior confining magnetic field is elliptic in the sense that (3.8) is associated with an elliptic operator.

5.1. *The case of elliptic jet flow*

When the jet flow is elliptic, β is real and positive. In this case, (5.1) can be solved for non-repeated roots by determining the non-tangential intersections between the two (real) continuous curves $Z = Z_1(z)$ and $Z = \lambda\beta Z_2(\beta z)$. Henceforth we assume that $z > 0$.

Now, since $K_1(z) > K_0(z) > 0$ and $I_0(z) > I_1(z) > 0$ (see, for example, appendix, equation (A 8)), from (5.2)

$$0 < Z_1(z) < 1, \quad Z_2(z) > 1. \tag{5.10}$$

Also, by the asymptotic formulae for the modified Bessel functions (Watson 1944),

$$Z_1(z) = 1-\frac{1}{2z}+\frac{3}{8z^2}+O\left(\frac{1}{z^3}\right) \to 1_- \quad \text{as} \quad z \to +\infty, \tag{5.11}$$

$$Z_2(z) = 1+\frac{1}{2z}+\frac{3}{8z^2}+O\left(\frac{1}{z^3}\right) \to 1_+ \quad \text{as} \quad z \to +\infty \tag{5.12}$$

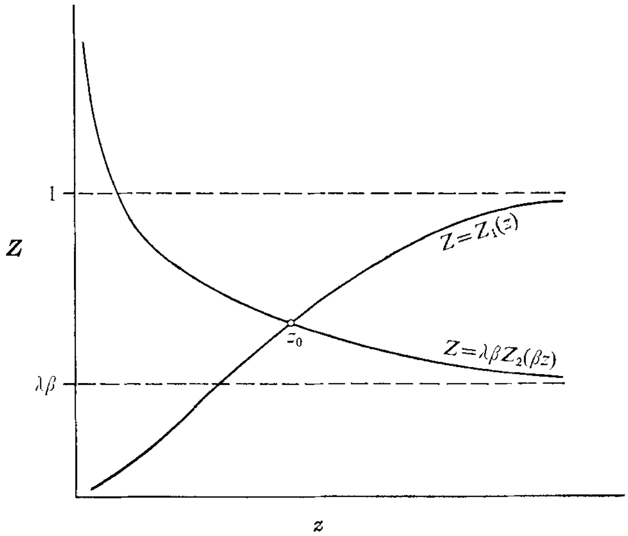

FIGURE 2. The case $\beta^2 > 0$, $0 < \lambda\beta < 1$.

and, furthermore, from their infinite (power) series in z,

$$Z_1(z) \to 0_+, \quad Z_2(z) \to +\infty \quad \text{as} \quad z \to 0_+. \tag{5.13}$$

Now, according to (A 1) (see appendix),

$$F_1(z) > z > F_2(z) \quad (> 0).$$

So (5.3) and (5.4) imply

$$Z_1'(z) > 0, \quad Z_2'(z) < 0, \tag{5.14}$$

that is, $Z_1(z)$ is monotonic increasing (with increasing z) while $Z_2(z)$ is monotonic decreasing. The properties (5.10)–(5.14) suffice to enable an appropriate sketch to be made of each of the curves $Z = Z_1(z)$ and $Z = \lambda\beta Z_2(\beta z)$ (see figure 2). Evidently, these two curves intersect each other if and only if

$$0 < \lambda\beta < 1, \tag{5.15}$$

there being only one possible intersection at $z = z_0$, say, and this is essentially non-tangential. [Note that a point of intersection between any two curves is solely represented here, and throughout the rest of this paper, by its abscissa, its ordinate being omitted for reasons of economy.] Taking (3.19) and (3.28) into account, the left inequality of (5.15) is, effectively, $M < 1$, $A > 1$, i.e. the jet flow must be subsonic and super-Alfvénic.

By virtue of (5.3), (5.4), (5.10) and (5.14), the square-bracketed numerator factor occurring in (5.8) is

$$\frac{\beta Z_2'(\beta r_0|\alpha_\nu|)}{Z_2(\beta r_0|\alpha_\nu|)} - \frac{Z_1'(r_0|\alpha_\nu|)}{Z_1(r_0|\alpha_\nu|)} < 0.$$

Thus, if the flow criterion $M < 1$, $A > 1$ is further narrowed to

$$M < 1, \quad M^2 + A^2 \geqslant 2, \tag{5.16}$$

then, in view of (5.8) and (5.9),

$$V(|\alpha_0|) < 0. \tag{5.17}$$

Here, $z_0 = r_0|\alpha_0|$ is the specified single point of intersection indicated in figure 2, and it exists provided that the accompanying inequality ($\lambda\beta < 1$) also holds. In this case, (4.12) and (4.13) reduce to

$$\xi(x,r) \sim -\frac{\chi_0(x)\, K_1(r|\alpha_0|)\, I_1(\beta r_0|\alpha_0|)}{r_0|\alpha_0|\, K_1(r_0|\alpha_0|)\, G_{|\alpha|}(|\alpha_0|, U)} H(-x) \quad (r \geqslant r_0), \tag{5.18}$$

$$\eta(x,r) \sim -\frac{\chi_0(x)\, I_1(\beta r|\alpha_0|)}{r_0|\alpha_0|\, G_{|\alpha|}(|\alpha_0|, U)} H(-x) \quad (r \leqslant r_0), \tag{5.19}$$

which are valid within $|x| \gg l$. Here, the functions $\chi_0(x)$ and $G_{|\alpha|}(|\alpha_0|, U)$ are obtainable from (4.14) and (5.7) respectively. Hence each perturbation asymptotically emerges as a single upstream wave with wavenumber $|\alpha_0|$. [Note that, for the jet-flow regime included within $M < 1$, $A > 1$ and adjacent to (5.16), i.e. $A > 1$, $M^2+A^2 < 2$, there is also a single wave function, provided, of course, that $\lambda\beta < 1$. However, it is generally difficult to determine from (5.8) whether such a wave should arise upstream or downstream of the radiating source.]

The remaining possible regimes of elliptic jet flow are

$$M > 1, \quad A < 1 \tag{5.20}$$

(corresponding to a supersonic sub-Alfvénic jet),

$$M^2+A^2 < 1, \tag{5.21}$$

and

$$M < 1, \quad A > 1 \quad \text{but} \quad \lambda\beta \geqslant 1. \tag{5.22}$$

Here, both (5.20) and (5.21) violate the left-hand inequality of (5.15) while the right-hand inequality is obviously violated by (5.22). It follows that (5.1) has no (real) root. Consequently, we conclude that, within $|x| \gg l$,

$$\xi(x,r) \sim 0, \quad \eta(x,r) \sim 0 \tag{5.23}$$

if the jet flow is governed by any one of the conditions (5.20)–(5.22).

In the case $M < 1$, $A > 1$ with $\lambda\beta < 1$, the interaction at the separating boundary $r = r_0$ apparently imparts to the entire elliptic configuration of motion of both the jet column and its external confining magnetic field a pseudo-hyperbolic element, giving rise to a (real) axially non-decaying wave. Nevertheless, this is a single wave function, unlike a genuine hyperbolic (wave) solution, which normally constitutes an infinite continuous (integral) superposition of real wave functions. In contrast, however, the initial states corresponding to (5.20)–(5.22) do conform to the ellipticity of motion in the sense that all perturbations diminish with increasing axial (as well as, in the case of the exterior region $r > r_0$, outward transverse) distances from the source.

5.2. *The case of hyperbolic jet flow*

Suppose that the jet flow is hyperbolic. Then $\beta = i|\beta|$ with

$$|\beta| = \left(\frac{(A^2-1)(M^2-1)}{M^2+A^2-1}\right)^{\frac{1}{2}}. \tag{5.24}$$

Again, let us assume throughout that $z > 0$. Hence, since

$$I_0(\beta z) = J_0(|\beta| z), \quad I_1(\beta z) = iJ_1(|\beta| z)$$

($J_0(z)$ and $J_1(z)$ being Bessel functions of the first kind), (5.1) becomes

$$Z_1(z) = \lambda|\beta| Z_3(|\beta| z), \tag{5.25}$$

where

$$Z_3(z) = J_0(z)/J_1(z). \tag{5.26}$$

Furthermore, if we define

$$F_3(z) = \frac{J_0(z) J_1(z)}{J_0^2(z) + J_1^2(z)}, \tag{5.27}$$

then (5.7) is expressible as

$$G_{|\alpha|}(|\alpha_\nu|, U) = i \operatorname{Re}[G_{|\alpha|}(|\alpha_\nu|, U)], \tag{5.28}$$

with

$$\operatorname{Re}[G_{|\alpha|}(|\alpha_\nu|, U)] = r_0 \lambda |\beta| K_1(r_0|\alpha_\nu|) J_0(|\beta| r_0|\alpha_\nu|) \left[\frac{|\beta|}{F_3(|\beta| r_0|\alpha_\nu|)} - \frac{1}{F_1(r_0|\alpha_\nu|)}\right]. \tag{5.29}$$

Also, from (5.8), the group velocity function

$$V(|\alpha_\nu|) = \frac{U[1/F_1(r_0|\alpha_\nu|) - |\beta|/F_3(|\beta| r_0|\alpha_\nu|)]}{2A^2/r_0|\alpha_\nu| (A^2-1) - \frac{1}{2}U(\partial|\beta|^2/\partial U)/|\beta| F_3(|\beta| r_0|\alpha_\nu|)}, \tag{5.30}$$

where (cf. equation (5.9)),

$$\frac{U}{2}\frac{\partial|\beta|^2}{\partial U} = \frac{M^2A^2(M^2+A^2-2)}{(M^2+A^2-1)^2}. \tag{5.31}$$

The (positive) distinct roots of (5.1), or equivalently of (5.25), will now be established by the non-tangential intersections between the curve $Z = Z_1(z)(\lambda|\beta|)^{-1}$ (whose essential features are already known via (5.10)–(5.14) or from figure 2), and the curve $Z = Z_3(|\beta| z)$. Relevant properties of the function $Z_3(z)$, defined by (5.26), are naturally deduced from those of $J_0(z)$ and $J_1(z)$. These two latter functions are continuous and oscillatory:

$$J_0(0) = 1, \qquad J_0(z) > 0 \quad \text{in} \quad (0, m_0),$$

$$J_0(z) < 0 \quad \text{in} \quad (m_{2\nu}, m_{2\nu+1}), \qquad J_0(z) > 0 \quad \text{in} \quad (m_{2\nu+1}, m_{2\nu+2}),$$

$$J_1(z) > 0 \quad \text{in} \quad (n_{2\nu}, n_{2\nu+1}), \qquad J_1(z) < 0 \quad \text{in} \quad (n_{2\nu+1}, n_{2\nu+2}),$$

where the sets of points

$$z = m_\nu, \qquad z = n_\nu \quad (\nu = 0, 1, \ldots, \infty),$$

are the two infinite discrete distributions of positive (distinct) zeros of $J_0(z)$ and $J_1(z)$ respectively (see Watson 1944, 15.2–15.22). In particular, $n_0 \equiv 0$. Also, the zeros at $z = m_\nu$ are interlaced with the zeros at $z = n_\nu$:

$$n_\nu < m_\nu < n_{\nu+1} < m_{\nu+1} \quad (\nu = 0, 1, \ldots, \infty).$$

Consequently, the set $Z = Z_3(z)$ is composed of an infinite sequence of continuous branch curves separated by vertical asymptotes passing through

$$z = n_\nu \quad (\nu = 1, 2, \ldots, \infty).$$

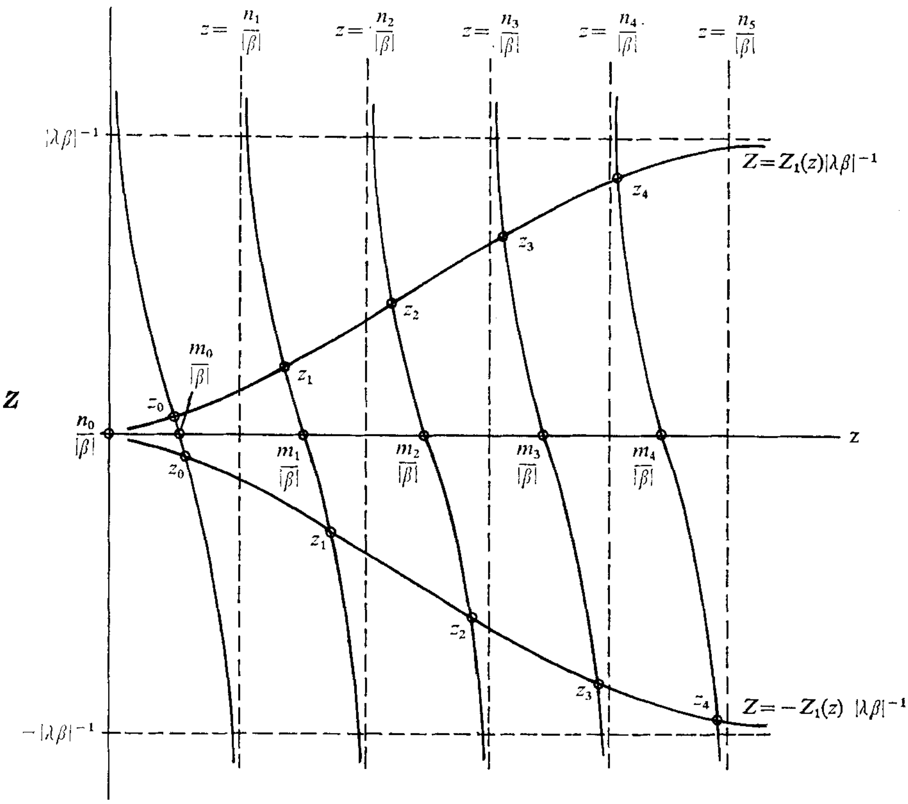

FIGURE 3. The case $\beta^2 < 0$. The branch curves, separated by the vertical asymptotes $z = n_\nu/|\beta|$ $(\nu = 1, 2, \ldots)$, constitute the set $Z = Z_3(|\beta| z)$ within $z > 0$. The intersections of this set with the set $Z = Z_1(z)\ (\lambda|\beta|)^{-1}$, in $z > 0$, are schematically represented for both $\lambda > 0$ and $\lambda < 0$.

The Z axis is also a vertical asymptote. Moreover,

$$Z_3(z) > 0 \quad \text{in} \quad (n_\nu, m_\nu), \qquad Z_3(z) < 0 \quad \text{in} \quad (m_\nu, n_{\nu+1}), \tag{5.32}$$

$$Z_3(z) = 0 \quad \text{at} \quad m_\nu \quad (\nu = 0, 1, \ldots, \infty), \tag{5.33}$$

$$Z_3(z) \to +\infty \quad \text{as} \quad z \to n_{\nu+} \quad (\nu = 0, 1, \ldots, \infty), \tag{5.34}$$

$$Z_3(z) \to -\infty \quad \text{as} \quad z \to n_{\nu-} \quad (\nu = 1, 2, \ldots, \infty). \tag{5.35}$$

Thus, the curve $Z = Z_3(|\beta| z)$ assumes the shape illustrated in figure 3 and, irrespective of whether $\lambda \gtrless 0$, always intersects the curve $Z = \pm Z_1(z)\ |\lambda\beta|^{-1}$ non-tangentially and an infinite number of times at the points $z = z_\nu$ $(\nu = 0, 1, \ldots, \infty)$. Since $\beta^2 = -|\beta|^2$, the condition $\lambda > 0$ becomes, in essence,

$$M < 1, \quad A < 1 \quad \text{but} \quad M^2 + A^2 > 1, \tag{5.36}$$

in which case, the points of intersection (see figure 3) satisfy

$$n_\nu < |\beta| z_\nu < m_\nu \quad (\nu = 0, 1, \ldots, \infty). \tag{5.37}$$

On the other hand, $\lambda < 0$ becomes equivalent to

$$M > 1, \quad A > 1, \tag{5.38}$$

and, under this condition,

$$m_\nu < |\beta|\, z_\nu < n_{\nu+1} \quad (\nu = 0, 1, \ldots, \infty). \tag{5.39}$$

The two inequalities (5.36) and (5.38) constitute the only possible hyperbolic jet-flow regimes.

Let us now focus attention upon the case when (5.38) holds, i.e. when the jet flow is supersonic and super-Alfvénic. The wave spectrum is infinite but discrete, with wavenumbers $|\alpha_\nu| = z_\nu/r_0$ governed by (5.39) and derived from the lower set of intersections (see figure 3) located below the z axis. Evidently, then, via (5.26) and (5.27),

$$F_3(|\beta|\, r_0|\alpha_\nu|) < 0,$$

and so (5.30) shows that $\quad V(|\alpha_\nu|) > 0 \quad (\nu = 0, 1, \ldots, \infty).$ (5.40)

So (4.12) and (4.13) yield, for $|x| \gg l$,

$$\xi(x,r) \sim H(x) \sum_{\nu=0}^{\infty} \frac{\chi_\nu(x)\, K_1(r|\alpha_\nu|)\, J_1(|\beta|\, r_0|\alpha_\nu|)}{r_0|\alpha_\nu|\; K_1(r_0|\alpha_\nu|)\, \mathrm{Re}\,[G_{|\alpha|}(|\alpha_\nu|, U)]} \quad (r \geqslant r_0), \tag{5.41}$$

$$\eta(x,r) \sim H(x) \sum_{\nu=0}^{\infty} \frac{\chi_\nu(x)\, J_1(|\beta|\, r|\alpha_\nu|)}{r_0|\alpha_\nu|\, \mathrm{Re}\,[G_{|\alpha|}(|\alpha_\nu|, U)]} \quad (r \leqslant r_0), \tag{5.42}$$

where $\mathrm{Re}\,[G_{|\alpha|}(|\alpha_\nu|, U)]$ is given by (5.29). In particular, every wave function in the spectrum is generated downstream. We also note, in passing, that inside the fluid column, each wave constituent of $\eta(x,r)$ possesses an oscillatory amplitude which vanishes intermittently at

$$r = n_j|\beta\alpha_\nu|^{-1} < r_0 \quad (j = 0, 1, \ldots),$$

where the n_j's are the zeros of $J_1(z)$.

For the complementary hyperbolic regime (5.36) there is also an infinite discrete wave spectrum, whose wavenumbers $|\alpha_\nu| = z_\nu/r_0$ comply with the inequality (5.37) and are obtained from the upper set of intersections (see figure 3) occurring above the z axis. However, it is difficult to determine the sign of $V(|\alpha_\nu|)$.

Being an infinite (though discrete) superposition of wave functions, each perturbation solution somewhat resembles the solution to a hyperbolic differential equation. Apparently, then, there is a tendency to emphasize the hyperbolic character of the interior fluid motion within $r < r_0$ and conceal the ellipticity of the exterior magnetic motion in $r > r_0$.

5.3. *The field-free jet*

In the absence of the internal trapped field $(H, 0, 0)$, we have $A = \infty$. Whence

$$\beta = (1 - M^2)^{\frac{1}{2}}, \quad \lambda = A_0^2/\beta^2. \tag{5.43}$$

Here, $A_0 = U/a_0$ (an Alfvén number of the interface).

Suppose that

$$M > 1, \tag{5.44}$$

corresponding to a supersonic jet flow which is evidently in the hyperbolic regime. In this case, (5.38) is invariably satisfied. So (5.39) and (5.40) hold, and (5.41) and (5.42) are typical asymptotic solutions encountered downstream.

Next, let us consider the flow field

$$M^2+A_0^4 < 1. \tag{5.45}$$

It follows that $M < 1$, i.e. that the jet flow is subsonic and belongs to the elliptic regime. Moreover, $0 < \lambda\beta < 1$, which is precisely the inequality (5.15). Hence the sole (real and non-repeated) wavenumber existing under the present circumstances is $|\alpha_0|$. Since (5.16) is an obvious consequence, (5.17) is valid. Single stationary wave functions are thus observed in the far upstream domain and have the forms (5.18) and (5.19).

Finally, if

$$M^2+A_0^4 \geqslant 1, \quad M < 1, \tag{5.46}$$

whereby the jet flow is again elliptic, but $\lambda\beta \geqslant 1$. The motion occurs under the restriction (5.22), and so is representable by the zero solutions (5.23).

The criteria (5.44)–(5.46) exhaust all relevant modes of motion in the vicinity of the field-free jet.

Appendix

The analysis of §5 leans heavily on a certain inequality, namely, that if $z > 0$ then

$$\frac{K_0(z)\,K_1(z)}{K_1^2(z)-K_0^2(z)} > z > \frac{I_0(z)\,I_1(z)}{I_0^2(z)-I_1^2(z)} > 0, \tag{A 1}$$

$K_n(z)$ and $I_n(z)$ being the modified Bessel functions. To prove this, we first appeal to the following Nicholson's representations (via Watson 1944, 13.72), valid within $|\arg z| < \frac{1}{2}\pi$:

$$K_0^2(z) = 2\int_0^\infty K_0(2z\cosh\phi)\,d\phi, \quad K_1^2(z) = 2\int_0^\infty K_2(2z\cosh\phi)\,d\phi, \tag{A 2}$$

$$K_0(z)\,K_1(z) = 2\int_0^\infty K_1(2z\cosh\phi)\cosh\phi\,d\phi; \tag{A 3}$$

$$I_0^2(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} I_0(2z\cos\phi)\,d\phi, \quad I_1^2(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} I_2(2z\cos\phi)\,d\phi, \tag{A 4}$$

$$I_0(z)\,I_1(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} I_1(2z\cos\phi)\cos\phi\,d\phi. \tag{A 5}$$

Hence, in view of the recurrence relations

$$K_2(z)-K_0(z) = (2/z)\,K_1(z), \quad I_0(z)-I_2(z) = (2/z)\,I_1(z),$$

(A 2) and (A 4) yield, respectively,

$$K_1^2(z)-K_0^2(z) = \frac{2}{z}\int_0^\infty \frac{K_1(2z\cosh\phi)}{\cosh\phi}\,d\phi, \tag{A 6}$$

$$I_0^2(z)-I_1^2(z) = \frac{2}{z\pi}\int_0^{\frac{1}{2}\pi} \frac{I_1(2z\cos\phi)}{\cos\phi}\,d\phi. \tag{A 7}$$

Now, when $z > 0$, we know that $K_n(z) > 0$ and $I_n(z) > 0$ $(n = 0, 1)$. So, in particular, (A 6) and (A 7) imply

$$K_1(z) > K_0(z) > 0, \quad I_0(z) > I_1(z) > 0 \quad (z > 0). \tag{A 8}$$

Finally, by coupling (A 3) with (A 6) and (A 5) with (A 7), we obtain, for $z > 0$,

$$K_0(z)\,K_1(z) - z[K_1^2(z) - K_0^2(z)] = 2\int_0^\infty K_1(2z\cosh\phi)\frac{\sinh^2\phi}{\cosh\phi}\,d\phi > 0, \quad \text{(A 9)}$$

$$z[I_0^2(z) - I_1^2(z)] - I_0(z)\,I_1(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} I_1(2z\cos\phi)\frac{\sin^2\phi}{\cos\phi}\,d\phi > 0, \quad \text{(A 10)}$$

from which, together with (A 8), the rule (A 1) follows.

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